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Maximum fidelity for a mirror symmetric set of qubit states

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Abstract
In this paper we address the problem of optimal reconstruction of a quantum state from the result of a single measurement, when the original quantum state is known to be a member of some specified set. This process provides both classical information about the state and a reproduction of that state. A suitable figure of merit for this process is the fidelity, which is the probability that the state we construct on the basis of the measurement result is found by a subsequent test to match the original state. We consider the maximization of the fidelity for a mirror symmetric set of three pure qubit states, but find that our results are more generally applicable. In contrast to previous examples, we find that the strategy which minimizes the probability of erroneously identifying the state does not generally maximize the fidelity.

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1. Introduction

The principles governing communication through a quantum channel have been extensively studied. The transmitting agent (Alice) selects a state from a predefined set $\{|\psi_j\rangle\}$ with relative frequency p_j and transmits a quantum system prepared in this state through the quantum channel. The classical information that is Alice's message is encoded on a string of such states. The receiving agent (Bob) knows the set of possible signal states $\{|\psi_j\rangle\}$ and their relative frequencies p_j. These p_j are the prior probabilities he assigns to the states before he makes a measurement. Bob must make a measurement on the states he receives to attempt to recover the encoded information.

The problem for Bob is to determine the best measurement to make. Which measurement is best will depend on how the results are to be used, that is how the information was encoded or what question about the signal states the measurement is designed to answer. For each such coding or question one can define a mathematical 'figure of merit' function which provides a measure of how appropriate a given measurement strategy is. Bob's task in finding his optimal measurement is to maximize this function with respect to all possible measurements. Commonly considered examples of this are the minimum error probability (or minimum Bayes cost) [1–4] and the accessible information [4–7], both of which describe recovery of classical information about the original message.

For some applications, Bob needs to use his measurement result to reproduce the quantum signal. The objective is then that the new signal matches the original as closely as possible. We must now consider optimal strategies for the combined measurement and reconstruction process. The quality of these measurement–retransmission strategies is associated with the *fidelity F*. This is the probability that a subsequent measurement on the retransmitted state will confirm that it matches the original signal. The fidelity is a measure of the amount of quantum information as opposed to classical information obtained from the signal.

One motivation for a discussion of the fidelity of these processes is the question of eavesdropping in quantum communication or cryptography, where an eavesdropper would require both classical information about the signal and a good reproduction of that signal to conceal their presence. Measurement and retransmission is by no means the only possible attack on cryptographic protocols.

The fidelity can also be used as a figure of merit for many other processes where it is desirable for the final state of a (real or hypothetical) system to match some ideal state as closely as possible. An important example is state estimation where one uses a measurement to determine the value of some continuous parameter(s) of the system. Fidelity is often used as the optimality criterion when the parameters to be measured are the unknown parameters of the density matrix of the system. In contrast we use the fidelity to describe how well we can reproduce a finite set of discrete states, by a dual process consisting of a measurement step followed by a reconstruction step.

It can be seen that estimation is a continuous variable analogue of our maximum fidelity reproduction problem, in the same way as minimum cost estimation [1] can be seen as the continuous version of optimal hypothesis testing [1]. Maximum fidelity estimation for the case where all possible values of the continuous parameters are equally probable has been studied [8]. This case is of particular relevance to the problem of optimal universal cloning [9–14]. In a similar fashion, the problem we consider in this paper, that of maximizing fidelity for a discrete set of possible states will be related to the question of state-dependent cloning [12, 14].

No general condition is known for maximizing the fidelity of a measurement and retransmission strategy for a known discrete set of signal states with arbitrary prior probabilities, but the maximum fidelity has been found for specific cases. These cases are when the possible signals form a set of symmetric qubit states [15], and where there are only two possible signal qubit states [16]. Here we will describe the maximum fidelity strategy for a mirror-symmetric set of three pure qubit states.

For the two previously solved cases [15, 16] the measurement strategy which minimizes the probability of incorrectly identifying the states always maximizes the fidelity for the best choice of retransmission states. This optimal strategy is not, however, unique for sets of three or more symmetric states. It is therefore interesting to ask whether the strategy that minimizes the error probability always maximizes the fidelity. If this is the case then our best strategy is to identify the original signal state as well as we can and then select a corresponding retransmission state. In this paper we establish that the fidelity is *not* always maximized by the measurement strategy which minimizes the probability of erroneously identifying the signal state. We demonstrate this by maximizing the fidelity for the mirror-symmetric qubit states, for which the minimum error strategy has recently been derived [17].

2. Fidelity

The previous work on maximum fidelity for symmetric states [15] established some important results which we shall make use of. We shall use the notation contained in that work. The signal states are denoted by $|\psi_j\rangle$ with associated prior probabilities p_j and the retransmission states are $|\phi_k\rangle$. The measurement is described by its probability operator measure (POM) elements $\hat{\Pi}_k$. These POM elements are operators which represent the probability of occurrence of each possible outcome of a measurement. The probability $P(k|j)$ of the outcome k occurring given that the system was prepared in the state $|\psi_j\rangle$ is

$$P(k|j) = \langle\psi_j|\hat{\Pi}_k|\psi_j\rangle. \tag{1}$$

For the POM elements $\hat{\Pi}_k$ to represent probabilities, they must be subject to the following constraints:

(i) All the $\hat{\Pi}_k$'s are Hermitian.
(ii) Their eigenvalues are non-negative.
(iii) The total probability of all outcomes for any input sums to 1:

$$\sum_k \hat{\Pi}_k = \hat{1}. \tag{2}$$

To find the optimal measurement–retransmission strategy we need to express the fidelity F as a function of the POM elements $\hat{\Pi}_k$, the retransmission states $|\phi_k\rangle$ and the set of possible signal states $\{|\psi_j\rangle, p_j\}$. The fidelity is the probability that the state $|\phi_k\rangle$ selected on the basis of the measurement outcome $\hat{\Pi}_k$ will pass a test of the question 'Is the state $|\psi_i\rangle$?'. This test is described by the POM $\{|\psi_i\rangle\langle\psi_i|, \hat{1} - |\psi_i\rangle\langle\psi_i|\}$, where $|\psi_i\rangle$ is the state of the original signal. Thus F is given by

$$F = \sum_{j,k} |\langle\psi_j|\phi_k\rangle|^2 \langle\psi_j|\hat{\Pi}_k|\psi_j\rangle p_j. \tag{3}$$

This can be written as [15]

$$F = \sum_k \langle\phi_k|\hat{O}_k|\phi_k\rangle \tag{4}$$

where the positive operator $\hat{O}_k$ is given by

$$\hat{O}_k = \sum_j |\psi_j\rangle\langle\psi_j|\hat{\Pi}_k|\psi_j\rangle\langle\psi_j| p_j. \tag{5}$$

It is clear from this that the optimal retransmission states $|\phi_k\rangle$ are the eigenvectors of the operators $\hat{O}_k$ corresponding to the largest eigenvalue ν_{k_+} of $\hat{O}_k$.

If these optimal retransmission states are used, then the fidelity is given by the sum of the largest eigenvalues of the $\hat{O}_k$ operators:

$$F = \sum_k \nu_{k_+} \tag{6}$$

and we need only consider the maximization of the largest eigenvalues of $\hat{O}_k$, subject to the constraint that the operators $\hat{\Pi}_k$ form a POM. In such a maximization each of the POM elements $\hat{\Pi}_k$ can be assumed to be proportional to a pure state projector, since the action of any mixed-state like element here would be identical to that of a number of rank 1 POM elements corresponding to the same retransmission state.

3. Mirror-symmetric state sets

We have recently described the minimum error strategy for a qubit which is known to be one of a mirror-symmetric set of three pure qubit states [17]. Here mirror symmetric means that the set of states $\{(|\psi_j\rangle, p_j)\}$ is invariant under the transformation

$$|+\rangle \longrightarrow +|+\rangle \qquad |-\rangle \longrightarrow -|-\rangle \tag{7}$$

so that the prior probability associated with any state is equal to the prior probability of its mirror-symmetric counterpart.

The set of three mirror-symmetric pure qubit states can be written in its most general form as

$$\begin{aligned} |\psi_1\rangle &= \cos\theta|+\rangle + \sin\theta|-\rangle \qquad && p_1 = p \\ |\psi_2\rangle &= \cos\theta|+\rangle - \sin\theta|-\rangle && p_2 = p \\ |\psi_3\rangle &= |+\rangle && p_3 = 1-2p \end{aligned} \tag{8}$$

where $0 \leqslant \theta \leqslant \frac{\pi}{2}$ and $0 \leqslant p \leqslant \frac{1}{2}$.

The minimum error strategy was found to be of different form in two distinct domains of p and θ. The solutions in these two domains are

for

$$p \geqslant \frac{1}{2+\cos\theta(\cos\theta+\sin\theta)} \tag{9}$$

the minimum error measurement strategy is given by

$$\hat{\Pi}_1 = \tfrac{1}{2}(|+\rangle + |-\rangle)(\langle +| + \langle -|) \qquad \hat{\Pi}_2 = \tfrac{1}{2}(|+\rangle - |-\rangle)(\langle +| - \langle -|) \qquad \hat{\Pi}_3 = 0 \tag{10}$$

and for

$$p \leqslant \frac{1}{2+\cos\theta(\cos\theta+\sin\theta)} \tag{11}$$

the minimum error measurement strategy is given by

$$\begin{aligned} \hat{\Pi}_1 &= \tfrac{1}{2}(a|+\rangle + |-\rangle)(a\langle +| + \langle -|) \\ \hat{\Pi}_2 &= \tfrac{1}{2}(a|+\rangle - |-\rangle)(a\langle +| - \langle -|) \\ \hat{\Pi}_3 &= (1-a^2)|+\rangle\langle +| \end{aligned} \tag{12}$$

where a is the following function of p and θ:

$$a = \frac{p\cos\theta\sin\theta}{1-p(2+\cos^2\theta)}. \tag{13}$$

At the boundary between the two domains, which is when the equality holds in the condition (9), $a = 1$ and thus $\hat{\Pi}_3 = 0$.

We must now find the maximum fidelity measurement strategy for these mirror symmetric states to show that it is different from the minimum error strategy.

4. Maximizing fidelity for the mirror-symmetric state sets

To find the maximum fidelity for these mirror-symmetric sets of states we will follow a similar method to that used for the symmetric states [15]. We attempt to write an explicit formula for the fidelity in terms of some parameter set and find the maximum by differentiation.

To maximize the fidelity for these mirror-symmetric sets of states we choose a representation of the operator $\hat{O}_k$ and find its eigenvalues. To do this we first obtain a general representation of the qubit POM elements. As we stated in section 2 we need only consider elements of rank 1, that is, elements proportional to pure state projectors.

The elements of such a POM can be represented by the matrices

$$\hat{\Pi}_k = w_k \begin{pmatrix} 1+\cos\theta_k & \sin\theta_k\,\mathrm{e}^{\mathrm{i}\phi_k} \\ \sin\theta_k\,\mathrm{e}^{-\mathrm{i}\phi_k} & 1-\cos\theta_k \end{pmatrix} \tag{14}$$

where the basis vectors are

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{15}$$

and $0 \leqslant w_k \leqslant \frac{1}{2}$, $-\pi < \theta_k \leqslant \pi$, $-\frac{\pi}{2} < \phi_k \leqslant \frac{\pi}{2}$.

These POM elements are automatically Hermitian and positive. The remaining completeness constraint (2) becomes equations for w_k, θ_k and ϕ_k:

$$1 - \sum_k w_k = 0 \tag{16}$$

$$\sum_k w_k \cos\theta_k = 0 \tag{17}$$

$$\sum_k w_k \sin\theta_k \mathrm{e}^{\mathrm{i}\phi_k} = 0. \tag{18}$$

The operators $\hat{O}_k$ for a mirror symmetric set of three pure states become

$$\hat{O}_k = w_k \begin{pmatrix} \begin{matrix} 2p\cos^2\theta(1+\cos 2\theta\cos\theta_k) \\ +(1-2p)(1+\cos\theta_k) \end{matrix} & p\sin^2 2\theta\sin\theta_k\cos\phi_k \\ p\sin^2 2\theta\sin\theta_k\cos\phi_k & 2p\sin^2\theta(1+\cos 2\theta\cos\theta_k) \end{pmatrix}. \tag{19}$$

The eigenvalues $\nu_{k_\pm}$ of this matrix are given by

$$\frac{\nu_{k_\pm}}{w_k} = p(1+\cos 2\theta\cos\theta_k) + \left(\frac{1}{2}-p\right)(1+\cos\theta_k) \pm \left[\left(p\cos 2\theta(1+\cos 2\theta\cos\theta_k) + \left(\frac{1}{2}-p\right)(1+\cos\theta_k)\right)^2 + p^2\sin^4 2\theta\sin^2\theta_k\cos^2\phi_k\right]^{\frac{1}{2}} \tag{20}$$

of which the greater eigenvalue is clearly ν_{k_+}.

From the form of these eigenvalues we see that the fidelity F is an even function of θ_k. This means that any element with the parameters (w_k, θ_k, ϕ_k) gives the same contribution to the fidelity as would an element with the parameters $(w_k, -\theta_k, \phi_k)$, and thus the same contribution as the pair of elements $\left(\frac{w_k}{2}, \theta_k, \phi_k\right)$ and $\left(\frac{w_k}{2}, -\theta_k, \phi_k\right)$. Thus we can replace all of the elements in a POM with such pairs of elements without changing the fidelity, and we need only find the maximum fidelity for POMs consisting of such pairs. Such POMs satisfy condition (18) automatically. Since there is now no condition restricting our choice of ϕ_k, we are free to select ϕ_k to maximize each eigenvalue ν_{k_+} independently. It is clear from examination of (20) that the best choice is always $\phi_k = 0$ and thus $\cos\phi_k = 1$. In truth we should have expected such a symmetry of our measurement strategy, since this simply corresponds to the POM also being both mirror symmetric and confined to the plane of the states $\{|\psi_j\rangle\}$.

Since the pair of elements corresponding to $\pm\theta_k$ are equally weighted and each contributes the same amount to the fidelity, we now use the parameter w_k to represent the combined weight of the pair of elements with the same value of $\cos\theta_k$.

We can then write the eigenvalues as

$$\nu_{k_+} = w_k \left[p\,(1 + \cos 2\theta \cos\theta_k) + \left(\tfrac{1}{2} - p\right)(1 + \cos\theta_k) + Q_k^{\frac{1}{2}} \right] \tag{21}$$

where the functions Q_k are given by

$$Q_k = \left[p \cos 2\theta\,(1 + \cos 2\theta \cos\theta_k) + \left(\tfrac{1}{2} - p\right)(1 + \cos\theta_k)\right]^2 + p^2 \sin^4 2\theta \sin^2\theta_k. \tag{22}$$

The POM constraints (16) and (17) allow us to simplify the fidelity F:

$$F = \sum_k \nu_{k_+} = \frac{1}{2} + \sum_k w_k Q_k^{\frac{1}{2}}. \tag{23}$$

To find the stationary points of F, subject to the constraints (16, 17) on w_k and θ_k we shall use Lagrange's method of undetermined multipliers. We can construct the function G:

$$G = F + \alpha_1 \left(1 - \sum_k w_k \right) + \alpha_2 \left(\sum_k w_k \cos\theta_k \right) \tag{24}$$

with the constraint (18) being irrelevant as this F does not depend on ϕ_k. The full detail of the maximization calculation can be found in appendix A, but the main points are summarised here.

The equation $\frac{\partial G}{\partial \theta_k} = 0$ has four solutions for θ_k: $\theta_k = 0, \pi$ and $\pm\Omega$, where Ω is some angle depending on the unknown multiplier α_2. This limits the number of elements in any optimal POM to four.

The minimum value of w_k, $w_k = 0$, clearly corresponds to trivial zero operators. The maximum value of w_k for a mirror symmetric pair of elements arises from the positivity of the POM elements and the completeness condition (2). These conditions impose a tight bound on $\langle +|\hat{\Pi}_k|+\rangle$ and $\langle -|\hat{\Pi}_k|-\rangle$, and thus on w_k. If this bound is reached, then there can be only one other element in the POM, which must be proportional to either $|+\rangle\langle +|$ or $|-\rangle\langle -|$ to satisfy the completeness condition. Thus, if the optimal measurement strategy is composed of more than three elements, then all of the elements must satisfy the equation $\frac{\partial G}{\partial w_k} = 0$ as well as $\frac{\partial G}{\partial \theta_k} = 0$.

It is possible to show that there are no values of the unknown multipliers α_1 and α_2 which simultaneously satisfy $\frac{\partial G}{\partial w_k} = 0$ for all four solutions of $\frac{\partial G}{\partial \theta_k} = 0$, except in special cases where F does not depend on θ_k (for $\cos\phi_k = 1$).

Having established that the optimal strategy consists of three or less elements we simplify the problem by applying the POM conditions (16) and (17) to obtain the most general three-element mirror-symmetric POM. This POM has only one free parameter, $\cos\Omega$, and it is now a simple matter to maximize F. The optimal measurement strategy is always found to be one or other of the two mirror-symmetric two-element strategies.

Obtaining the optimal retransmission states is simply a matter of finding the eigenvector of $\hat{O}_k$ corresponding to the larger of its two eigenvalues. The determination of these states is detailed in appendix B.

5. Complete maximum fidelity strategy

We summarize the results for the measurement–retransmission strategy maximizing the fidelity for a mirror symmetric set of three pure states.

If

$$p(1 - \cos 2\theta)[p(1 - \cos 2\theta) + \cos 2\theta] = 0 \tag{25}$$

then any POM consisting of elements of the form

$$\hat{\Pi}_k = w_k \begin{pmatrix} 1 + \cos\theta_k & \sin\theta_k \\ \sin\theta_k & 1 - \cos\theta_k \end{pmatrix} \tag{26}$$

which satisfies POM conditions (16)–(18) with $\phi_k = 0$ will maximize the fidelity. The optimal retransmission state for each element is then given by

$$|\phi_k\rangle = \left(Y_k{}^2 + 1\right)^{-\frac{1}{2}} (|+\rangle + Y_k|-\rangle) \tag{27}$$

with Y_k given by equation (B.2).

If

$$p < -\frac{\cos 2\theta}{1 - \cos 2\theta}, \qquad p \neq 0 \tag{28}$$

then the unique optimal measurement strategy consists of the two elements

$$\hat{\Pi}_0 = |+\rangle\langle+| \qquad \hat{\Pi}_\pi = |-\rangle\langle-| \tag{29}$$

and the optimal retransmission state is $|+\rangle$ if the result is $\hat{\Pi}_0$ and $|-\rangle$ if the result is $\hat{\Pi}_\pi$.

If

$$p > -\frac{\cos 2\theta}{1 - \cos 2\theta}, \qquad p \neq 0, \qquad \cos 2\theta \neq 1 \tag{30}$$

then the unique optimal measurement strategy consists of the two elements

$$\hat{\Pi}_{+\frac{\pi}{2}} = \tfrac{1}{2}(|+\rangle + |-\rangle)(\langle+| + \langle-|) \qquad \hat{\Pi}_{-\frac{\pi}{2}} = \tfrac{1}{2}(|+\rangle - |-\rangle)(\langle+| - \langle-|) \tag{31}$$

with the optimal retransmission state for these elements given by

$$\left|\phi_{\pm\frac{\pi}{2}}\right\rangle = \left[1 + \left(\sqrt{\eta^2 + 1} - \eta\right)^2\right]^{-\frac{1}{2}} \left[|+\rangle \pm \left(\sqrt{\eta^2 + 1} - \eta\right)|-\rangle\right] \tag{32}$$

where η is given by

$$\eta = \frac{2p\cos 2\theta + 1 - 2p}{2p\sin^2 2\theta}.$$

6. Comments on the optimal strategy

Where there is a unique maximum fidelity solution, the optimal measurement always has two elements and is both invariant under the mirror symmetry (7) and confined to the plane of the signal states. There are only two such POMs and when one gives maximum fidelity, the other gives the least fidelity of all POMs confined to the plane of the Bloch sphere containing the states. Where these two measurements give the same fidelity condition (25) holds, there is no unique optimal measurement and any POM composed of elements in the plane of the states will be optimal.

Condition (25) is obviously satisfied when there is only one possible signal state ($p = 0$) and when all of the signal states are identical ($\theta = 0$). It is also satisfied if the sum of the density matrices of the states, each multiplied by the probability of *not* selecting that state, sums to the identity matrix, as described by the equation

$$\sum_j (1 - p_j)|\psi_j\rangle\langle\psi_j| = \hat{1}. \tag{33}$$

The importance of this final condition remains unclear, but the simplicity of (33) suggests that it may have physical significance for our problem.

The retransmission states $|\phi_k\rangle$ for the $\theta_k = 0, \pi$ solution are simple to understand as they are just the states that the POM elements project on to. The origin of the retransmission states for the $\theta_k = \pm\frac{\pi}{2}$ case seems less transparent, but they are simply the states that the POM elements project on to, rotated to increase their overlap with the *a priori* state of the signal.

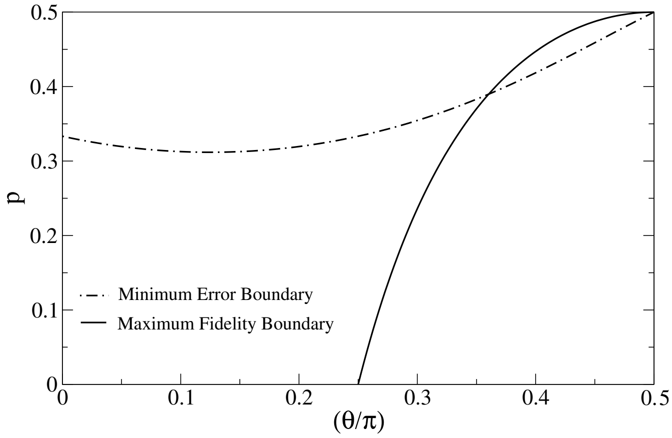

Figure 1. A comparison of the domains in which the $\theta_k = \pm\frac{\pi}{2}$ strategy is optimal for minimum error (equation (9), above the dashed-dotted line) and for maximum fidelity (equation (30), to the left of the solid line), in terms of the state parameters p and θ.

The hypothesis we sought to test, that the fidelity is always maximized by the minimum error measurement strategy, must now be examined. It is clearly false as the $\theta_k = 0, \pi$ is never a minimum error strategy, while it is often a unique maximum fidelity strategy. Furthermore, the three-element minimum error strategy never maximizes the fidelity, save when any strategy confined to the plane of the signal states maximizes the fidelity.

This leaves us to consider the $\theta_k = \pm\frac{\pi}{2}$ strategy. It gives minimum error uniquely when (9) is true and maximum fidelity uniquely when (30) is true. However, we can see from figure 1 that the conditions (9) and (30) bear little relation to each other, except that both are satisfied for large enough p. Thus we can say that the maximum fidelity strategy is not the same as, or even dependent on, the minimum error strategy.

There are two special cases of the previous work in [15] which are also covered by our calculation here. Our work is in complete agreement with [15] in predicting that the maximum fidelity strategy for two equiprobable states $\left(p = \frac{1}{2}\right)$ is the same as the minimum error strategy, and that any POM confined to the plane of the signal states will be optimal for the trine states $\left(p = \frac{1}{3}, \theta = 60^\circ\right)$. It is interesting to note that we predict the same solution for a much wider class of state sets than just the trine states, so it may be possible to extend the solution for the general set of symmetric states to a broader class of sets.

Finally, in the course of the solution for a set of three mirror-symmetric states we only referred to the nature and number of the states themselves in finding the eigenvalues of our $\hat{O}_k$ matrices. The only property of these eigenvalues that we used in deriving our solution was that their sum depended on θ_k and ϕ_k only via the functions $\cos\theta_k$ and $\cos^2\phi_k$. The rest of the analysis was done using the coefficients of $\cos\theta_k$ (A, B and C in the appended calculations) and holds true for any form of these coefficients. This implies that for *any* set of qubit states for which this eigenvalue sum has a similar dependence only on $\cos\theta_k$ and $\cos^2\phi_k$, the optimal measurement strategy will again be either $\theta_k = \left\{\pm\frac{\pi}{2}\right\}$ or $\theta_k = \{0, \pi\}$ (with $\cos\phi_k = 1$), or any set of elements in the linear plane if the fidelity of these two strategies is the same. In particular this means our strategy maximizing the fidelity for a mirror-symmetric set of three pure states is also the measurement strategy for maximizing the fidelity for a mirror-symmetric set of any number of pure states sharing a common plane. The form of the condition (A.26) will obviously become more complex in terms of the original variables when there are more states, but will be unchanged as a function of the coefficients of $\cos\theta_k$ (A.21).

7. Conclusions

The fidelity of a measurement and reconstruction strategy is defined as the average probability that a subsequent measurement on the reconstructed state will identify it as being identical to the original state. We sought to find the strategy which maximizes the fidelity for a signal, the state of which is known to be a member of a mirror-symmetric set of three qubit states. This was done by parametrizing the POM, evaluating the fidelity as a function of these parameters and conducting a variational calculation using Lagrange's method of undetermined multipliers to identify the sets of elements which could constitute the maximum.

The optimal measurement strategy was found to be whichever of the two mirror-symmetric two-element POMs gives the larger fidelity for a given set of states, and that any POM whose elements lie in the plane of the states is optimal if these two strategies give the same fidelity. The optimal retransmission states were found using an eigenvector equation (B.1).

This result also holds for a mirror-symmetric set of *any* number of pure qubit states which are located on the same great circle of the Bloch sphere.

Unlike all previous solutions for maximum fidelity strategies, the minimum error measurement strategy does not generally maximize the fidelity of a mirror-symmetric set of three pure states.

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Appendix A. Derivation of the fidelity maximizing measurement

In attempting to maximize the fidelity F subject to POM conditions (16) and (17) it is helpful to begin by using Lagrange's method of undetermined multipliers. We construct the function G:

$$G = F + \alpha_1 \left(1 - \sum_k w_k\right) + \alpha_2 \left(\sum_k w_k \cos\theta_k\right). \tag{A.1}$$

Varying G with respect to w_k gives us a restriction on the possible positions of the global maxima and minima of G, which must be located either at a stationary point with respect to w_k:

$$\frac{\partial G}{\partial w_k} = Q_k^{\frac{1}{2}} - \alpha_1 + \alpha_2 \cos\theta_k = 0 \tag{A.2}$$

or at the maximum or minimum possible values of w_k:

$$w_{k:\min} = 0 \qquad w_{k:\max} = \frac{1}{1 + |\cos\theta_k|} \tag{A.3}$$

with this maximum arising from the fact that w_k represents the combined weight of the pair of POM elements corresponding to $\pm\theta_k$, and that the component of such a pair in either the $|+\rangle$ or $|-\rangle$ direction cannot exceed 1. The minimum value of w_k clearly corresponds to trivial zero operators.

Since Q_k is a function of $\cos\theta_k$ only, differentiating G with respect to θ_k gives

$$\frac{\partial G}{\partial\theta_k} = \frac{w_k}{2} Q_k^{-\frac{1}{2}} \frac{\partial Q_k}{\partial \cos\theta_k}(-\sin\theta_k) - \alpha_2 w_k \sin\theta_k = 0 \tag{A.4}$$

at a stationary point of G. This has the non-trivial solutions

$$\sin\theta_k = 0 \Rightarrow \theta_k = 0 \text{ or } \pi \tag{A.5}$$

and

$$\frac{\partial Q_k}{\partial \cos\theta_k} = -2\alpha_2 Q_k^{\frac{1}{2}} \tag{A.6}$$

as well as the $w_k = 0$ solution corresponding to the element not being part of the POM.

To simplify further analysis we shall assign the coefficients of $\cos\theta_k$ in Q_k to be

$$A = p\cos 2\theta + \left(\tfrac{1}{2} - p\right)$$
$$B = p\sin^2 2\theta = \tfrac{1}{2} - C \geqslant 0$$
$$C = p\cos^2 2\theta + \left(\tfrac{1}{2} - p\right) = \tfrac{1}{2} - B \geqslant 0$$

so that Q_k is given by

$$Q_k = (A + C\cos\theta_k)^2 + B^2(1 - \cos^2\theta_k).$$

We can then solve equation (A.6) to find the remaining values of θ_k which satisfy $\frac{\partial G}{\partial\theta_k} = 0$. Since equation (A.6) contains only $\cos\theta_k$ terms, it is simplest to express the solution as a value of $\cos\theta_k$ given in terms of our A, B, C coefficients as

$$\cos\theta_k = \frac{1}{C^2 - B^2}\left(-AC - \alpha_2 B\sqrt{\frac{A^2 + B^2 - C^2}{\alpha_2^2 + B^2 - C^2}}\right) \tag{A.7}$$

which can only take one value for a given POM since α_2 must have a single value for all of the elements of one POM.

We can now say that any measurement maximizing the fidelity has at most four elements, corresponding to the four solutions of $\frac{\partial G}{\partial\theta_k} = 0$ (A.4) for θ_k, given by equations (A.5) and (A.7). For each of these possible elements, either equation (A.2) holds (a stationary point of G with respect to w_k) or w_k takes its maximum or minimum value.

Now we must consider whether it is possible to have all four of these elements present in one POM, i.e. that none of the weight factors w_k are zero for these elements. Clearly this implies that none of them take their maximum values either, since the maximum value of w_k for an element (or mirror-symmetric pair of elements) is found by noting that the positivity and completeness of the POM implies that neither the $|+\rangle\langle+|$ or $|-\rangle\langle-|$ component of any element (or pair) can exceed 1. If any element saturates this bound, there can be only *one* more non-zero element corresponding to either $|+\rangle\langle+|$ or $|-\rangle\langle-|$ to satisfy the completeness condition (2).

Since no weight factor w_k can attain its maximum value when there are four non-zero elements present in the POM, the equation (A.2) must be simultaneously satisfied for all four elements. This occurs if there is a pair of values for α_1 and α_2 which will satisfy (A.2) for all three solutions for $\cos\theta_k$ obtained from (A.4).

For $\theta_k = 0$ and π, equation (A.2) gives

$$\theta_k = 0 : |A + C| = \alpha_1 - \alpha_2 \tag{A.8}$$

$$\theta_k = \pi : |A - C| = \alpha_1 + \alpha_2 \tag{A.9}$$

which fixes both α_1 and α_2 for any measurement strategy containing both these elements. These values must satisfy the equation (A.2) for the value of $\cos\theta_k$ given by (A.7). Since the multipliers α_1 and α_2 can only take the values $\pm A$, $\mp C$, respectively, or $\pm C$, $\mp A$, respectively, for any values of A and C when (A.8) and (A.9) hold, it is simple to show that equation (A.2) can only be satisfied for this value of $\cos\theta_k$ when either

$$A^2 + B^2 - C^2 = 0 \qquad \text{or} \qquad B = 0. \tag{A.10}$$

Which of these two conditions is relevant depends on the relative magnitudes and signs of A and C. Thus we see that we can only have a four-element POM in certain special cases.

Examining these special cases shows that in each of them Q_k is the square of some linear function of $\cos\theta_k$ which is either positive for all θ_k or negative for all θ_k. If $\left|Q_k^{\frac{1}{2}}\right|$ is any linear function of $\cos\theta_k$, it can be shown by application of the POM conditions (16) and (17) that F does not depend on any $\cos\theta_k$ and thus the fidelity is constant for any measurement strategy composed of elements confined to the plane of the states (that is for which $\cos\phi_k = 1$).

For the general case where F does depend on the strategy chosen we now know that there is no strategy composed of four or more elements which can be a maximum or minimum of the fidelity. Denoting the solution of equation (A.7) in the range $0 \leqslant \theta_k \leqslant \pi$ as $\theta_k = \Omega$, there are two possibilities for three element strategies: case (i)

$$\theta_k = \pi, \pm\Omega \qquad 1 \geqslant \cos\Omega \geqslant 0 \tag{A.11}$$

or case (ii)

$$\theta_k = 0, \pm\Omega \qquad -1 \leqslant \cos\Omega \leqslant 0. \tag{A.12}$$

Both the two element measurement strategies which are mirror symmetric in this basis are special cases of these three-element strategies, and are located at the edge of the domains of the three-element strategies. The $\theta_k = \{0, \pi\}$ strategy corresponds to $\cos^2\Omega = 1$ in either of the above cases, and the $\theta_k = \left\{\pm\frac{\pi}{2}\right\}$ strategy corresponds to $\cos\Omega = 0$.

For these three-element strategies the POM conditions (16) and (17) place a strict limit on the values of the weight factors w_k. Denoting the weights of the $\theta_k = 0, \pi$ elements and the $\theta_k = \pm\Omega$ pair by w_0, w_π and w_Ω, respectively, we have either

(i) for $\cos\Omega \geqslant 0$

$$w_\pi + w_\Omega = 1 \qquad w_\pi - w_\Omega\cos\Omega = 0 \tag{A.13}$$

which gives

$$w_\Omega = \frac{1}{1+\cos\Omega} \qquad w_\pi = \frac{\cos\Omega}{1+\cos\Omega} \tag{A.14}$$

or

(ii) for $\cos\Omega \leqslant 0$

$$w_0 + w_\Omega = 1 \qquad w_0 + w_\Omega\cos\Omega = 0 \tag{A.15}$$

which gives

$$w_\Omega = \frac{1}{1-\cos\Omega} \qquad w_0 = \frac{-\cos\Omega}{1-\cos\Omega}. \tag{A.16}$$

Now we need simply differentiate F with respect to $\cos\Omega$ for each of these two strategies and select the largest value of F from any stationary points and the two limiting two-element strategies. Both of these strategies automatically satisfy all of the POM conditions so we no longer need to use Lagrange's method of undetermined multipliers.

Case (i). For the $\theta_k = \pi, \pm\Omega$ case the stationarity equation $\frac{\partial F}{\partial\cos\Omega} = 0$ can be rearranged and squared to obtain

$$B^2(C^2 - B^2 - A^2)(1+\cos\Omega)^2 = 0. \tag{A.17}$$

The only solutions to this equation are the aforesaid special cases (A.10) and $\cos\Omega = -1$, which is not allowed since $\cos\Omega \geqslant 0$ in this case. We conclude that there are no stationary points of F for this set of strategies and the maximum and minimum of the fidelity for these strategies must correspond to the two-element strategies which define the end points of our variation (i.e. $\cos\Omega = 0$ or 1).

Case (ii). Similarly for the $\theta_k = 0, \pm\Omega$ case, $\frac{\partial F}{\partial \cos\Omega} = 0$ implies that

$$B^2(C^2 - B^2 - A^2)(1 - \cos\Omega)^2 = 0. \tag{A.18}$$

As before, the only solutions to this are our two special cases (A.10) and the single value of $\cos\Omega = 1$, which is not in the domain for this strategy. We can thus conclude that there are no stationary points of F for either strategy and our global maximum and minimum must correspond to the $\theta_k = \left\{\pm\frac{\pi}{2}\right\}$ and $\theta_k = \{0, \pi\}$ strategies which are at the end points of both of our three-element strategy domains.

The fidelity for each of the two strategies which must constitute our maximum and minimum are

- for $\theta_k = \left\{\pm\frac{\pi}{2}\right\}$

$$F = \tfrac{1}{2} + \sqrt{B^2 + A^2} \tag{A.19}$$

- for $\theta_k = \{0, \pi\}$

$$F = \frac{1}{2} + \frac{|A + C|}{2} + \frac{|A - C|}{2} \tag{A.20}$$

which is just equal to a half plus the larger of $|A|$ or $|C|$.

The larger of these two fidelities, (A.19) and (A.20), will be the maximum fidelity, and the corresponding POM will be the optimal measurement strategy. For the $\theta_k = \{0, \pi\}$ strategy to be optimal, we must have $C \geqslant |A|$ since the fidelity of the $\theta_k = \left\{\pm\frac{\pi}{2}\right\}$ strategy is always at least $\frac{1}{2} + |A|$. Thus the $\theta_k = \{0, \pi\}$ strategy is only uniquely optimal if

$$A^2 + B^2 - C^2 < 0 \tag{A.21}$$

which can be restated in terms of the original variables p and θ as

$$p < -\frac{\cos 2\theta}{1 - \cos 2\theta}. \tag{A.22}$$

The two strategies give the same fidelity when the relevant condition in (A.10) holds. This corresponds to the special case where any POM consisting of elements confined to the plane of the states is optimal. These conditions can be written in terms of p and θ as

$$p(1 - \cos 2\theta)[p(1 - \cos 2\theta) + \cos 2\theta] = 0 \tag{A.23}$$

that is either

$$p = 0 \tag{A.24}$$

or

$$\cos 2\theta = 1 \tag{A.25}$$

or

$$p = -\frac{\cos 2\theta}{1 - \cos 2\theta}. \tag{A.26}$$

The meaning of two of these three cases is clear: equation (A.24) is the case where there is only one possible signal state and equation (A.25) describes the case where all three states are identical. The fidelity obviously cannot depend on the measurement strategy at all in these cases. The third of these cases, equation (A.26), is less obvious. In fact it corresponds to the identity

$$\sum_j (1 - p_j)|\psi_j\rangle\langle\psi_j| = \hat{1} \tag{A.27}$$

that is when the sum of the density operators of the states normalized to the prior probability of *not* selecting that state is the identity operator.

Appendix B. Retransmission states

In equation (4) we found the optimal retransmission states to be the eigenvectors of $\hat{O}_k$, which is given by equation (5). This determines the best retransmission state for any measurement we choose to make, not only the optimal measurement. The optimal retransmission state $|\phi_k\rangle$ depends on the possible states of the original signal $\{|\psi_j\rangle\}$ and on the direction of the *corresponding* measurement operator $\hat{\Pi}_k$. Since $|\phi_k\rangle$ does not depend on the weight (w_k) of this element or on the rest of the POM, $|\phi_k\rangle$ will be the optimal retransmission state for any POM containing an element in this direction. This is useful as we need only find $|\phi_k\rangle$ for each possible element, without having to consider the strategy in which the element occurs. It could therefore be said that the optimal retransmission state $|\phi_k\rangle$ depends on the result of the measurement (given by the direction of $\hat{\Pi}_k$) rather than on the measurement strategy (that is the experiment whose outcome was k).

It is simplest to find the states $\{|\phi_k\rangle\}$ if we consider the following three cases separately: $\theta_k = 0$, $0 < |\theta_k| < \pi$ and $\theta_k = \pi$.

For $\theta_k = 0$ our $\hat{O}_k$ matrix has eigenvectors $|+\rangle$ and $|-\rangle$. The larger eigenvalue belongs to $|+\rangle$ if $A + C > 0$. Since we found previously that we must have $C \geqslant |A|$ for $\theta_k = 0, \pi$ to be the best strategy, it is always the case that the optimal retransmission state for the element Π_0 is $|+\rangle$ when we have employed the optimal measurement strategy.

Similarly, for $\theta_k = \pi$ our $\hat{O}_k$ matrix again has eigenvectors $|+\rangle$ and $|-\rangle$. The larger eigenvalue now belongs to $|-\rangle$ if $A - C < 0$, which is again always true when the optimal strategy includes a $\theta_k = \pi$ element.

For any other value of θ_k, such as $\theta_k = \pm\frac{\pi}{2}$, we must find the general solution for the eigenvector corresponding to the larger eigenvalue of a 2×2 real Hermitian matrix. We need only study nondiagonal matrices since the $\hat{O}_k$ matrix is diagonal when $\theta_k \neq 0$ or π only for trivial sets of states $\{|\psi_j\rangle, p_j\}$. The equation for the unnormalized eigenvectors is

$$\begin{pmatrix} R_k & S_k \\ S_k & P_k \end{pmatrix} \begin{pmatrix} 1 \\ Y_{k\pm} \end{pmatrix} = \nu_{k\pm} \begin{pmatrix} 1 \\ Y_{k\pm} \end{pmatrix} \tag{B.1}$$

which gives a value for $Y_{k\pm}$ in terms of the matrix elements

$$Y_{k\pm} = \frac{P_k - R_k}{2S_k} \pm \frac{\sqrt{(P_k - R_k)^2 + 4S_k^2}}{2S_k} \tag{B.2}$$

where the $\pm$ in $Y_{k\pm}$ in this equation corresponds to the two eigenvalues $\nu_{k\pm}$, so the eigenvector of interest is that which contains Y_{k+}.

From the general form of $\hat{O}_k$ given in equation (19) we can identify the elements of our eigenvector equation as

$$
\begin{aligned}
R_k &= 2p\cos^2\theta(1+\cos 2\theta\cos\theta_k) + (1-2p)(1+\cos\theta_k) \\
P_k &= 2p\sin^2\theta(1+\cos 2\theta\cos\theta_k) \\
S_k &= p\sin^2 2\theta\sin\theta_k
\end{aligned}
\tag{B.3}
$$

from which we can identify Y_{k+} and thus find and normalize $|\phi_k\rangle$. It can be readily appreciated that the form of these states is not simple.

For the case where $\theta_k = \pm\frac{\pi}{2}$, we identify the parameter η as

$$
\eta = \frac{R_k - P_k}{2|S_k|} = \frac{2p\cos 2\theta + (1-2p)}{2p\sin^2 2\theta} \tag{B.4}
$$

then $Y_{\pm\frac{\pi}{2}}$ is given by

$$
Y_{\pm\frac{\pi}{2}} = \pm\left(\sqrt{\eta^2+1} - \eta\right). \tag{B.5}
$$

The optimal retransmission state for this strategy is given by

$$
\left|\phi_{\pm\frac{\pi}{2}}\right\rangle = \frac{1}{\sqrt{1+\left(Y_{\pm\frac{\pi}{2}}\right)^2}}\left[|+\rangle + \left(Y_{\pm\frac{\pi}{2}}\right)|-\rangle\right]. \tag{B.6}
$$

It is clear from the form of $Y_{\pm\frac{\pi}{2}}$ (B.5) that the retransmission state given by (B.6) is, as expected, on the same side of the Bloch sphere as the corresponding POM element $\hat{\Pi}_{\pm\frac{\pi}{2}}$. Further analysis of the physical meaning of these states is possible by rewriting the η parameter as

$$
\eta = \frac{\langle +|\hat{\rho}_T|+\rangle - \langle -|\hat{\rho}_T|-\rangle}{2p\sin^2 2\theta} \tag{B.7}
$$

where $\hat{\rho}_T$ is the state Bob assigns to the signal before making his measurement and is given by

$$
\hat{\rho}_T = \sum_j p_j |\psi_j\rangle\langle\psi_j|. \tag{B.8}
$$

It is clear that $\hat{\rho}_T$ is also a measure of the 'average state' of the signal sent by Alice and that η is positive if the $|+\rangle\langle +|$ component of $\hat{\rho}_T$ is larger than the $|-\rangle\langle -|$ component (the 'average state' is closer to $|+\rangle$ than $|-\rangle$) and negative if the converse is true. Furthermore we see that $\left|Y_{\pm\frac{\pi}{2}}\right|$ is larger than one if η is negative and smaller than one if η is positive, so that the retransmission states $\left|\phi_{\pm\frac{\pi}{2}}\right\rangle$ are shifted from $|+\rangle \pm |-\rangle$ towards the 'average state', $\hat{\rho}_T$.

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